Note on the Four Components of the Generalized Lorentz Group $O(n, 1)$

Kahar El-Hussein
Department of Mathematics, Faculty of Arts Science
Al Quryyat, Al-Jouf University, KSA
Department of Mathematics, Faculty of Science,
Al Furat University, Dear El Zore, Syria
E-mail : kumath@ju.edu.sa, kumath@hotmail.com

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Abstract

In physics and mathematics, the generalized Lorentz group is the group of all Lorentz transformations of generalized Minkowski spacetime $\mathbb{R}^{n+1}$. The Lorentz group is named for the Dutch physicist Hendrik Lorentz. The mathematical form of the kinematical laws of special relativity, Maxwell's field equations in the theory of electromagnetism, the Dirac equation in the theory of the electron, are each invariant under the Lorentz transformations. Therefore the Lorentz group is said to express the fundamental symmetry of many of the known fundamental Laws of Nature. Our purpose is to study the components of the general linear group $GL(n, \mathbb{R})$ and the four components of the generalized Lorentz group $O(n, 1)$. The identity component of $GL(n, \mathbb{R})$, denoted by $GL_+(n, \mathbb{R})$ is group and the identity component of $O(n, 1)$, denoted by $SO_+(n, 1)$ is group. We prove the negative components of the groups $GL(n, \mathbb{R})$ is group isomorphic onto $GL_+(n, \mathbb{R})$, and the remaining three components of the group $O(n, 1)$ are also groups each one of them isomorphic onto $SO_+(n, 1)$

Keywords: General Linear Group $GL(n, \mathbb{R})$, Generalized Lorentz Group $O(n, 1)$, New Groups

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1 Introduction.

In physics, the Lorentz transformation is named after the Dutch physicist Hendrik Lorentz. It was the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the laws of electromagnetism. The Lorentz transformation is in accordance with special relativity, but was derived well before special relativity. The transformations describe how measurements of space and time by two observers are related. They reflect the fact that observers moving at different velocities may measure different distances, elapsed times, and even different orderings of events. They supersede the Galilean transformation of Newtonian physics, which assumes an absolute space and time. The Galilean transformation is a good approximation only at relative speeds much smaller than the speed of light. The Lorentz transformation is a linear transformation. It may include a rotation of space; a rotation-free Lorentz transformation is called a Lorentz boost. In the Minkowski space, the Lorentz transformations preserve the spacetime interval between any two events. They describe only the transformations in which the spacetime event at the origin is left fixed, so they can be considered as a hyperbolic rotation of Minkowski space. The more general set of transformations that also includes translations is known as the Poincaré group. The Lorentz group is a subgroup of the Poincaré group, the group of all isometries of Minkowski spacetime. The Lorentz transformations are precisely the isometries which leave the origin fixed. Thus, the Lorentz group is an isotropy subgroup of the isometry group of Minkowski spacetime. For this reason, the Lorentz group is sometimes called the homogeneous Lorentz group while the Poincaré group is sometimes called the inhomogeneous Lorentz group. Lorentz transformations are examples of linear transformations; general isometries of Minkowski spacetime are affine transformations. Mathematically, the Lorentz group may be described as the non connected generalized orthogonal group $O(n,1)$, the matrix Lie group which preserves the quadratic form

$$(t, x_1, x_2, ..., x_n) \rightarrow t^2 - \sum_{i=1}^{n} x_i^2 \tag{1}$$

on $\mathbb{R}^{n+1}$. This quadratic form is interpreted in physics as the metric tensor of generalized Minkowski spacetime, so this definition is simply a restatement of the fact that Lorentz transformations are precisely the linear transformations which are also isometries of Minkowski spacetime $\mathbb{R}^{n+1}$. The Lorentz group is a $\frac{n(n+1)}{2}$-dimensional noncompact non-abelian real Lie group which is not connected. All four of its connected components are not simply connected. The identity component (i.e. the component containing the identity element) of the Lorentz group is itself a group and is often called the restricted Lorentz group and is denoted $SO_+(n,1)$. In this paper we will prove that the remaining three components of the group $O(n,1)$ have structure of group each one of them isomorphic onto the group $SO_+(n,1)$. To prove this, we begin by the general linear group $GL(n,\mathbb{R})$, which is not connected but rather has two connected components: the matrices with positive determinant and the ones $GL_-(n,\mathbb{R})$ with negative determinant. The identity component, denoted by $GL_+(n,\mathbb{R})$, consists of the real $n\times n$ matrices with positive determinant is group, and $GL_-(n,\mathbb{R})$ has structure of group isomorphic onto $GL_+(n,\mathbb{R})$.

2 New groups

Let $\mathbb{R}^*$be the multiplicative group and let $\mathbb{R}^*_+ = \{x \in \mathbb{R}^*;\ x \succ 0\}$ be the multiplicative group consists of all positive real numbers. Let $\mathbb{R}^*_- = \{x \in \mathbb{R}^*;$ $x \prec 0\}$ be the set consists of negative numbers

Definition 2.1. *On the set $\mathbb{R}^*_-$we define a new multiplication as*

$$x \bullet y = (-1)x.y \in \mathbb{R}^*_- \tag{2}$$

for any $x \lessdot 0$ and $y \lessdot 0$

Theorem 2.1. $(\mathbb{R}^*_-, \bullet)$ *with this law is group isomorphic onto the group* $\mathbb{R}^*_+$.

Proof: (i) the identity element is -1 because $x \bullet (-1) = (-1) \bullet x = x$.

(ii) The inverse element of x is $(-\frac{1}{x})$ due $x \bullet (-\frac{1}{x}) = (-1)x.(-\frac{1}{x}) = x$.

(iii) $x \bullet y = (-1)x.y = (-1)y.x = y \bullet x$ that means $\bullet$ is commutative

(iv) The associativity results immediately because $(x \bullet y) \bullet z = (-1)(x \bullet y).z = (-1)((-1)(x.y)).z = x.y.z = x \bullet (y \bullet z)$

(v) The mapping $\phi : \mathbb{R}^*_- \to \mathbb{R}^*_+$ defined by

$$\phi(x) = (-1)x \tag{3}$$

This implies

$$\phi(x \bullet y) = (-1)(x \bullet y) = (-1)(-1)x.y = x.y = \phi(x).\phi(y) \tag{4}$$

So ϕ is group homomorphism. It is clear ϕ is one-to-one and onto.

2.2. Let $(\mathbb{R}^*)^n = \mathbb{R}^* \times \mathbb{R}^* \times ... \times \mathbb{R}^*$ be the multiplicative group of dimension n, which is the direct product $n-$times of the multiplicative group $\mathbb{R}^*$ of dimension 1. Let $(\mathbb{R}^*_+)^n = \{(x_1, x_2, ..., x_n) \in (\mathbb{R}^*)^n | \ x_i \succ 0, \forall \ 1 \leq i \leq n \ \}$ be the multiplicative group of positive component and let $(\mathbb{R}^*_-)^n = \{(x_1, x_2, ..., x_n) \in (\mathbb{R}^*)^n | \ x_i \prec 0, \forall \ 1 \leq i \leq n \ \}$ be the negative component. As in definition 2.1 we furnish this component by new law noted again $\bullet$ as follows

$$x \bullet y = [(x_1, x_2, ..., x_n) \bullet (y_1, y_2, ..., y_n)] = [(-1)x_1.y_1, (-1)x_2.y_2, ..., (-1)x_n.y_n] \tag{5}$$

It is easy to show $((\mathbb{R}^*_-)^n, \bullet) = (\mathbb{R}^*_- \times \mathbb{R}^*_- \times ... \times \mathbb{R}^*_-, \bullet)$ is group and is the direct product $n-$times of the multiplicative group $\mathbb{R}^*_-$. Since $(\mathbb{R}^*_-, \bullet)$ *is group isomorphic onto the group* $\mathbb{R}^*_+$, then $((\mathbb{R}^*_-)^n, \bullet) = ((\mathbb{R}^*_+)^n, .)$ and so the non connected Lie group is two copies of $t$he group $((\mathbb{R}^*_+)^n, .)$

The general linear $GL(n, \mathbb{R})$ is not connected but rather has two connected components: the matrices with positive determinant is group contains the identity element and denoted by $GL_+(n, \mathbb{R})$ and the ones with negative determinant denoted by $GL_-(n, \mathbb{R})$. The orthogonal group $O(n)$ is maximal compact non connected subgroup of $GL(n, \mathbb{R})$, while the maximal compact subgroup of $GL_+(n, \mathbb{R})$ is the special orthogonal group $SO(n)$. We denote the component of the negative determinant by $GL_-(n, \mathbb{R})$ and we denote by $O_-(n)$ the subset of $O(n)$with negative determinant. Let $I_- \in GL_-(n, \mathbb{R})$ be the matrix defined as

$$I_- = (a_{ij}) \tag{6}$$

where $a_{11} = -1$, $a_{ii} = 1$ for any $2 \leq i \leq n$, and otherwise $a_{ij} = 0$.

Definition 2.2. *On the subset* $GL_-(n, \mathbb{R})$, *we put the multiplicative law* $\top$,*which is defined by*

$$A \top B = A.I_-.B \tag{7}$$

for any A *and* B *belong to* $GL_-(n, \mathbb{R})$

Theorem 2.2. $(GL_-(n, \mathbb{R}), \top)$*becomes group isomorphic onto the group* $(GL_+(n, \mathbb{R}),.)$

Proof: As in theorem 2.1, we can easily find that the identity element is I_-, because

$$I_- \top A = I_- = A \top I_- = I_-.I_-A = A.I_-.I_- = A \tag{8}$$

Since

$$A \top B = B \top A = A.I_-.B = B.I_-.A = I_- \tag{9}$$

then the inverse element of an element $A \in GL_-(n, \mathbb{R})$ is $I_-A^{-1}I_-$. The associativity holds because we have,

$$\begin{aligned} (A \quad \top \quad B) \top C &= A \top (B \top C) \\ &= (A \top B).I_-.C = (A.I_-.B).I_-.C \\ &= A.I_-.(B.I_-.C) = A \top (B \top C) \end{aligned} \tag{10}$$

for any A, B, and C,belong $GL_-(n, \mathbb{R})$. Now define mapping $\psi : GL_-(n, \mathbb{R}) \rightarrow GL_+(n, \mathbb{R})$ by

$$\psi(A) = AI_- \tag{11}$$

then we get

$$\psi(A \top B) = (A \top B).I_- = A.I_-.B.I_- = \psi(A).\psi(B) \tag{12}$$

That means ψ is group homomorphism and clearly is bijective. So our theorem is proved.

Corollary 2.1. *The set $O_-(n)$ with the law $\top$ becomes group isomorphic onto $SO(n)$*

The proof of this corollary results immediately from theorem 2.2.

3 The Generalized Lorentz Group $O(n, 1)$

3.1. The pseudo orthogonal groups $O(p, q)$ are of considerable interest in theoretical physics. It can be identified with the Lie group of all real $n \times n$ -matrices A which satisfy

$$A^t I_{p,q} A = I_{p,q} \tag{13}$$

where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad (14)$$

The most important are the Lorentz group $O(n,1)$ and the inhomogeneous de Sitter group $O(4,1)$. As in $[16, 105-109]$, we consider the four connected components of the group $O(n,1)$, which are corresponding to the cases

(1) All matrices X of the form

$$L_1 = \{X \in O(n,\mathbb{R});\ detX\langle 0;\ +1 \text{ as the lower right entry of the orthogonal matrix}\} \quad (15)$$

(2) All matrices X of the form

$$L_2 = \{X \in SO(n,\mathbb{R});\ -1 \text{ as the lower right entry of the orthogonal matrix}\} \quad (16)$$

(3) All matrices X of the form

$$L_3 = \{X \in O(n,\mathbb{R});\ detX\langle 0;\ -1 \text{ as the lower right entry of the orthogonal matrix}\} \quad (17)$$

(4) all matrices X of the form

$$L_4 = \{X \in SO(n,\mathbb{R});\ +1 \text{ as the lower right entry of the orthogonal matrix} \quad (18)$$

Thus (3) and (4) correspond to the group $SO(n,\ 1)$. Observe that $\det X = -1$ in the cases (1) and (2) and that $\det X = 1$ in the cases (3) and (4). The components in (1) and (2) are not groups and the case (4) corresponds to a group, which is the connected component of the identity and it is denoted $SO_0(n,\ 1)$. The following theorem proves us that the components (1),(2) and (3) of the group $O(n,1)$ are naturally equipped by structures of groups.

Theorem 3.1. The cases (1), (2) and (3) can be supplied by structure of groups, and each of them is isomorphic onto the group $SO_0(n,\ 1)$.

Proof : For the case (1), define the multiplication of two matrix X and Y belong to LD_1 by

$$X \diamond Y = X.I^-_{n+1}.Y \quad (19)$$

where

$$I^-_{n+1} = (\alpha_{ij}) \quad (20)$$

and $\alpha_{11} = -1, \alpha_{ii} = 1$,for $2 \leq i \leq n+1$, otherwise $\alpha_{ij} = 0$. Consider the equation

$$X \diamond Y = X.I^{-}_{n+1}.Y = Y \tag{21}$$

then the identity matrix is I^{-}_{n+1}, because

$$I^{-} \bullet X = X \bullet I^{-} = I^{-}.I^{-}.X = X.I^{-}.I^{-} = X \tag{22}$$

Since

$$X \diamond Y = X.I^{-}_{n+1}.Y = I^{-}_{n+1} \tag{23}$$

then the inverse X^{-1} of X is

$$I^{-}.X^{-1}.I^{-} \tag{24}$$

because

$$X \diamond X^{-1} = X.I^{-}_{n+1}.I^{-}.X^{-1}.I^{-} = I^{-}_{n+1} = X^{-1} \diamond X \tag{25}$$

The law $\bullet$ is associative, because

$$A \bullet (B \bullet C) = A.I^{-}.(B.I^{-}.C) = (A.I^{-}.B).I^{-}.C = (A \bullet B) \bullet C \tag{26}$$

and so $(LD_1, \bullet)$ becomes group. Now define the mapping $\Psi : LD_1 \rightarrow LD_4$ by

$$\Psi(X) = X.I^{-}_{n+1} \tag{27}$$

We have

$$\Psi(X \diamond Y) = (X \diamond Y).I^{-}_{n+1} = (X.I^{-}_{n+1}.Y).I^{-}_{n+1} = \Psi(X).\Psi(Y) \tag{28}$$

It is clear that the mapping Ψ is bijective and so is group isomorphism from LD_1 onto LD_4.

For the case (2), it suffices to define the multiplication in LD_2 by

$$X \triangleleft Y = X.I_{-,n+1}.Y \tag{29}$$

where

$$I_{-,n+1} = (\alpha_{ij}) \tag{30}$$

and $\alpha_{n+1n+1} = -1, \alpha_{ii} = 1$, for $1 \leq i \leq n$, otherwise $\alpha_{ij} = 0$. Obviously LD_2 with $\triangleleft$ turn into group and the mapping $\varphi : LD_2 \rightarrow LD_4$ which is defined by

$$\varphi(X) = X.I_{-,n+1} \tag{31}$$

is group isomorphic from LD_2 onto LD_4

For the case (3), define the multiplication law of LD_3 by

$$X \triangleright Y = X.I^{-}_{-,n+1}.Y \tag{32}$$

where

$$I^{-}_{-,n+1} = (\alpha_{ij}) \tag{33}$$

and $\alpha_{11} = -1$, $\alpha_{n+1n+1} = -1, \alpha_{ii} = 1$, for $2 \leq i \leq n$, otherwise $\alpha_{ij} = 0$. Clearly $\triangleright$ turn out LD_3 to group and the mapping $\varphi : LD_3 \to LD_4$ which is defined by

$$\Phi(X) = X.I^{-}_{-,n+1} \tag{34}$$

is group isomorphic from LD_3 onto LD_4. So the theorem is proved

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